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Taylor-Couette flow of ferrofluid: spin field and spin boundary condition effects

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Abstract

We solve for the steady flow solutions of a ferrofluid between concentric cylinders (the Taylor-Couette problem) and consider the effects of the spin field and the spin boundary conditions on the flow. In particular, our model includes the full spin equations. We analyze families of solutions for a range of realistic flow parameters and radial magnetic fields. Particular attention is paid to regimes in which different spin boundary conditions lead to significantly different flow profiles.

Keywords: Magnetic fluid; Taylor-Couette; steady shear; spin;

1. Introduction

Magnetic fluids are unique in that their properties, most notably viscosity, and their flows may be adjusted and controlled using modest magnetic fields. This feature has led to a wide array of ferrofluid based applications, from their well established use as high pressure seals and cooling fluids to their more recent role in micro-fluidic devices and in biomedicine.

In the development and design of new applications and devices, analytic descriptions of ferrofluid flows are typically based on one of the common approximate forms of the ferrofluid governing equations. Most commonly, the dynamic equation governing the internal angular momentum (or spin) is either neglected or used in a highly approximate form. For example, the stability studies in [1, 5, 6, 12, 13, 15, 17] all make simplifying assumptions on the spin equation. While such approximations can be justified using scaling arguments, an asymptotic analysis easily shows that the neglect of spin viscosity constitutes a singular perturbation, and therefore must be considered more carefully.

In a prior study [4] we investigated the impact of the spin viscosity on steady-state solutions of two-dimensional channel flow of magnetic fluid. In that work, we found that not only does the spin boundary layer influence the bulk

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flow profile, but that the numerical procedure to find exact solutions required extra care.

A critical obstacle to the adoption of a complete dynamic spin equation is the uncertainty in the form of the spin boundary condition. Good physical arguments have been made for both “zero spin” and “zero asymmetric stress” conditions on the spin at a boundary. In our earlier work on channel flow, we adopted only “zero asymmetric stress” conditions.

In the present work we turn to a more experimentally practical shear flow: the Taylor-Couette (TC) flow between concentric cylinders. In ordinary fluids [16], this flow has been used to explore both steady flows, as a viscometric model, and has also become a paradigm for the study of flow bifurcations. As such, it provides a better framework in which to examine the impact of the spin boundary layer on the steady flow solutions and on their bifurcation to instability. Vislovich and co-authors [17] first studied the linear stability of standard TC flow for a ferrofluid in a narrow gap between a fixed outer cylinder and a rotating inner cylinder and subject to a magnetic field having both radial and axial components. They show that the fields stabilize the TC flow. Niklas [5] considers the particular case of an imposed radial field and finds more accurate numerical solutions in this case. Stiles [14, 15] and co-authors consider the linear stability of ferro TC flows under radial fields and include effects of radial temperature gradients and particle diffusions. These effects are important in applications of ferrofluids to O-ring seals. Odenbach and co-authors [6, 7] consider the linear stability of ferro TC flow subject to an azimuthal field and point out that this setup can be used to probe for the rotational viscosity in a ferrofluid. Further, they mention experiments that disagree with theory and attribute this to possible chaining of the magnetic particles in the ferrofluid. Chang [1] and coworkers perform a broad parameter study of linear stability for ferro TC flow, while Ito [2] and Kikura [3] use the ultrasound method to experimentally measure the velocity profile of steady and unstable ferro TC flows. All of the linear stability studies mentioned above consider only perturbations in the “fluid” variables and leave the magnetic quantities unperturbed. Singh and Bajaj [12, 13] consider perturbations of the magnetic quantities in their stability analysis. All the linear stability studies cited here find that the magnetic fields stabilize the ferro TC flow. Also, as we have already mentioned, all of these studies either neglect or use a highly simplified form of the dynamic equation governing the spin.

In the following section we describe the governing equations and boundary conditions of the ferrofluid in the cylindrical Taylor-Couette geometry. We then outline our assumptions leading to the steady flow equations for radially directed applied fields and describe the numerical method used to obtain solutions. These solutions are presented and analyzed for physically realistic fields and for a range of control parameters.

2. Governing equations and boundary conditions

We consider a ferrofluid between two concentric cylinders of radii $R_1 < R_2$ separated by a distance $R = R_2 - R_1$ as shown in figure 1. A magnetic field $\mathbf{H}$ is applied along the radial direction. Thus, the inner cylinder acts as one pole of a magnet while the outer cylinder is the other pole. The governing equations for linear momentum and angular momentum balance and the magnetization equation are (see, for example [8, 9, 11]):

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = \nabla\cdot \underset{\approx}{\mathrm{T}} + \mathbf{f} \tag{1}$$

$$\rho I\left(\frac{\partial \omega}{\partial t} + \mathbf{u}\cdot\nabla\omega\right) = \nabla\cdot \underset{\approx}{\mathrm{C}} + \mathbf{T} + \mathbf{l} \tag{2}$$

$$\frac{\partial \mathbf{M}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{M} = \omega\times\mathbf{M} - \frac{1}{\tau}\left(\mathbf{M} - \mathbf{M}_0\right) \tag{3}$$

where bold faced quantities represent vectors and the quantities $\underset{\approx}{\mathrm{T}}$ and $\underset{\approx}{\mathrm{C}}$ are tensors; I is the moment of inertia density; $\mathbf{u}$, the translational velocity; ω, the spin velocity of the suspension; $\mathbf{M_0}$, the equilibrium magnetization; τ, the effective magnetization relaxation time and the other symbols have their usual meanings. For incompressible ferrofluids, the Cauchy stress tensor $\underset{\approx}{\mathrm{T}} = -p\underset{\approx}{\mathrm{I}} + \eta\left(\nabla\mathbf{u} + \nabla\mathbf{u}^T\right) + \zeta\underset{\approx}{\mathrm{E}}\cdot\left(\nabla\times\mathbf{u} - 2\omega\right)$, where η and ζ are the coefficients of shear and vortex viscosity; $\mathbf{T} = -\underset{\approx}{\mathrm{E}} : \underset{\approx}{\mathrm{T}}$ is the antisymmetric vector of the Cauchy stress; the couple stress tensor

$\underset{\approx}{C} = \eta'\left(\nabla\omega + \nabla\omega^T\right)$, where η' is the shear coefficient of spin viscosity; the body force density $\mathbf{f} = \mu_0 \mathbf{M}\cdot\nabla\mathbf{H}$, where μ_0 is the permeability of free space; and the body couple density $\mathbf{l} = \mu_0 \mathbf{M}\times\mathbf{H}$, the equilibrium magnetization for low magnetic fields is proportional to $\mathbf{H}, \mathbf{M_0} = \chi\mathbf{H}$, with a constant of proportionality χ depending on temperature and the composition of the suspension. The magnetic flux **B** is related to the magnetic field and magnetization by $\mathbf{B} = \mu_0(\mathbf{H}+\mathbf{M})$.

We assume a cylindrical (r, θ, z) coordinate system, and restrict all field quantities to be functions of r only. For the geometry in Figure 1, the velocity $\mathbf{u} = (0, u(r), 0)$ and the spin velocity $\omega = (0, 0, \omega(r))$. Moreover, Gauss's law for the magnetic flux and Ampere's law for the magnetic field in conjunction with zero jumps in the tangential component of **H** and the normal component of **B** yield **B** = ($B_1(r)$, $B_2(r)$, 0), **M** = ($M_1(r)$, $M_2(r)$, 0) and **H** = ($h(r)$, 0, 0). We assume that the applied magnetic field $h(r) = H/r$ is inversely proportional to the radial distance.

The magnetization equation (3) yields

$$M_1(r) = \frac{\chi_0 h(r)}{1+\tau^2\left(\frac{u(r)}{r} - \omega(r)\right)^2} \quad \text{and} \quad M_2(r) = \frac{-\chi_0 h(r)\left(\frac{u(r)}{r} - \omega(r)\right)}{1+\tau^2\left(\frac{u(r)}{r} - \omega(r)\right)^2} \tag{4}$$

for the components of the magnetization.

We introduce dimensionless quantities by means of the following scalings: $\tilde{\mathbf{u}} = \mathbf{u}/U$, $\tilde{\mathbf{H}} = \mathbf{H}/H$, $\tilde{\mathbf{M}} = \mathbf{M}/H$, $\tilde{\omega} = \omega/\Omega$, $\tilde{p} = p/\Pi$, *and* $\tilde{t} = (R/U)t$ for time, where the gap-width R also scales all lengths and $0 \le \tilde{r} \le 1$ is defined as $r = \tilde{r}R + R_1$. We impose the additional scaling assumption that equates the relaxation, spin, and fluid velocity timescales: $\tau = 1/\Omega = R/U$,and work with the dimensionless variables, dropping all tildes (˜).

The first component of the balance of linear momentum (1) gives that the pressure gradient across the flow is balanced by the gradient of the magnetic field and the centrifugal force. This equation is not used in our analysis. The second component of the linear momentum balance (1) reduces to

$$0 = (1+\alpha)\left\{u''(r) + \frac{R}{\bar{r}}u'(r) - \frac{R^2}{\bar{r}^2}u(r)\right\} - 2\alpha\omega'(r) - \beta h^2(r)\left\{\frac{u(r)R^2 - \omega(r)\bar{r}R}{\bar{r}^2 + \left[u(r)R - \omega(r)\bar{r}\right]^2}\right\} \tag{5}$$

and the balance of angular momentum (2) becomes

$$0 = \varepsilon\left\{\omega''(r) + \frac{R}{\bar{r}}\omega'(r)\right\} - 2\omega(r) + u'(r) + \frac{R}{\bar{r}}u(r) + \frac{\beta}{\alpha}\bar{r}h^2(r)\left\{\frac{u(r)R - \omega(r)\bar{r}}{\bar{r}^2 + \left[u(r)R - \omega(r)\bar{r}\right]^2}\right\} \tag{6}$$

where $\bar{r} = Rr + R_1$ with $0 \le r \le 1$, $\varepsilon = \eta'/R^2\zeta$, $\alpha = \zeta/\eta = 3\phi/2$, and $\beta = \mu_0\chi_0 H^2/\zeta$, where ϕ is the volume fraction of magnetic particles in the suspension. The variable $R_1 \le r \le R_2$ measures the ferrofluid distance from the axis, while the small parameter ε is proportional to the square of L_p/R, where L_p is the magnetic particle size in the fluid. The dimensionless parameter β is a ratio of magnetic and viscous forces. Using the scaling assumption $\tau = 1/\Omega = R/U$ we replace τ and U and retain only the dimensionless quantities Ω and R.

We assume no-slip boundary conditions $u(0) = \Omega R_1$ and $u(1) = 0$ corresponding to a rotating inner cylinder while the outer cylinder is stationary. In addition, we impose vanishing anti-symmetric stress boundary conditions for $\omega(r)$. Beginning with the vanishing anti-symmetric stress condition $2\omega = \nabla \times \mathbf{u}$, and using the dimensionless quantities introduced above yields $2\omega(r) = \frac{\partial u}{\partial r}(r) + \frac{u(r)}{r + R_1/R}$ at r = 0, 1 as the boundary conditions for spin. In this paper, we solve the coupled second order differential equations (5) and (6) as boundary value problems on $r \in [0,1]$ for the fields $u(r)$ and $\omega(r)$ with ϵ small but nonzero for ferrofluid flows influenced by r-dependent magnetic fields $h(r)$ and driven by a rotating inner cylinder while the outer cylinder is fixed. All results presented in the next section are for $\varepsilon \approx 10^{-6}$. The two-point boundary value problem is solved using MATLAB's *bvp4c* routine, a collocation based solver (see, for example [10]). To compute accurate and reliable solutions when ε is small, we perform a continuation in successively smaller values ε.

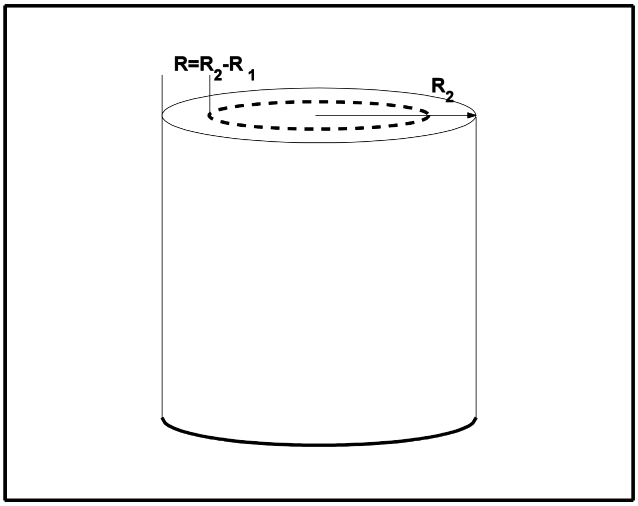

Figure 1: Taylor-Couette Setup: Ferrofluid between two cylinders of radii $R_1 < R_2$ in relative rotation with angular velocities Ω_1 and Ω_2 and gap-width $R = R_2 - R_1$. The magnetic field is radial.

3. Results and discussion

In this work we focus on the modification of the "ordinary" steady Taylor-Couette as a result of retaining the spin profile solution $\omega(r)$. In particular, we vary the spin equation parameters and the consequent spin boundary layer to produce steady ferrofluid Taylor-Couette flow profiles. In both figures, the black curves correspond to the fluid velocity $u(r)$ while the red curves represent the spin $\omega(r)$. We assume $\alpha = \beta = 0.15$, $R_1 = 1$, and $H = 2$ for the results presented below. In figure 2 we investigate the effect of changing the gap-width of the Taylor-Couette channel. When the gap width is narrow, $u(r)$ is almost linear while the spin velocity is constant in the channel. In this narrow gap regime, the external magnetic field effects dominate everywhere and the vorticity effects are almost constant resulting in strong boundary layer effects in $\omega(r)$. As the channel gap increases, magnetic field and vorticity effects are increasingly weaker away from the inner boundary. For large gaps, strong boundary layer effects in $\omega(r)$ are seen at the inner boundary while these are absent at the outer boundary. In figure 3 we present results when R and Ω are varied while the product velocity scale $R\Omega = U$ is constant. As expected, when the channel width is reasonably narrow (R = 0.5 and Ω = 2) and the rotation of the inner boundary is large, $\omega(r)$ has strong boundary layer effects near both boundaries. On the other hand, for larger channel widths (R = 2 and Ω = 0.5) and slower rotation rates, curvature effects begin to affect the spin profile.

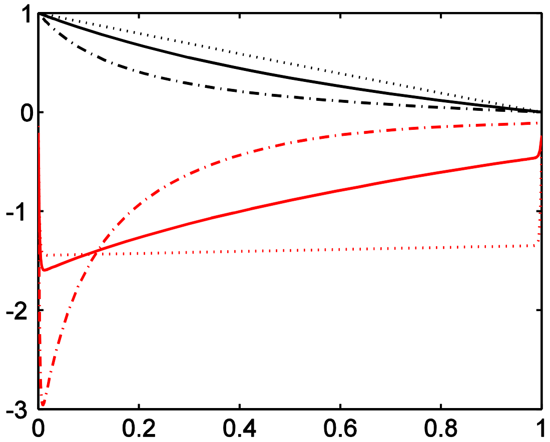

Figure 2: Gap-width study. Narrow gap $R = 0.05 \cdots$, medium gap $R = 1$ —,and large gap $R = 5$ − .−

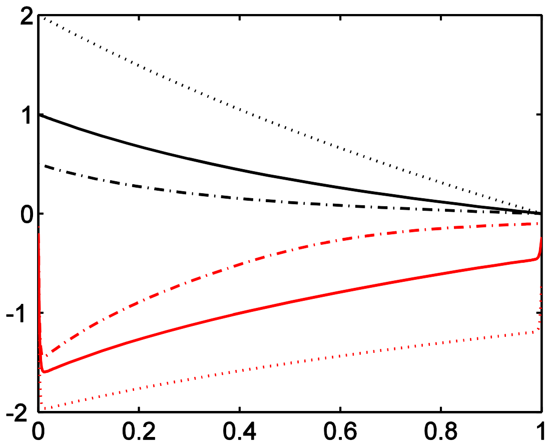

Figure 3: Constant fluid velocity-scale studies. $R = 0.5, \Omega = 2 \cdots$, $R = \Omega = 1$—, and $R = 2, \Omega = 0.5$ − .−

4. Conclusions

We have used a robust numerical method to find steady solutions for the velocity and spin fields of a ferrofluid between concentric cylinders (the Taylor-Couette problem) where the spinviscosity is assumed to be small but non-vanishing. We expect this to be a first step in a stability study for TC ferrofluid flows where the coupling between the fluid and spin velocities are retained. These results clearly show the competing effects of applied field and fluid vorticity on the steady velocity and spin in ferrofluid TC flow. These effects are also coupled to the spin boundary condition; it is possible that a carefully designed experiment would be able to deduce the actual physical boundary condition on spin indirectly, by measuring the velocity profile.

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